MATHCAD Program
using Gauss's Method to determine the Orbit
of an object from three measurements.

(Version 16
with speed of
light correction and one
iteration)

I work in time units of solar days, in distance units of AU (earth orbit radius), and masses of one solar mass (M). Then astronomers like to use the Gaussian gravitational constant k. In addition, I use mass units of 1 solar mass M.

$k := 0.01720209895$ $\qquad M0 := 1$ $\qquad i := 1..3$

The coordinates of three different comet measurements are the heliocentric vectors 1, 2, and 3. I start with Right Ascension and Declination measured here on Earth. I use α for right ascension angle, and δ for the declination.

$\Xi := 2\cdot\frac{\pi}{360}$ Data is entered in decimal degrees, and converted to radians.

I use time units of Julian dates, and then multiply them by the Gaussian gravitational constant k, to make the time come out in units of mean solar days.

$\alpha_1 := 264.0625\cdot\Xi$ $\qquad \delta_1 := -6.3402777\cdot\Xi$ $\qquad t_1 := 331.6667$ $\qquad$ Sept. 4, 1996 10:00 pm

$\alpha_2 := 263.766666\cdot\Xi$ $\qquad \delta_2 := -3.8586111\cdot\Xi$ $\qquad t_2 := 379.5833$ $\qquad$ Oct. 22, 1996 8:00 pm

$\alpha_3 := 271.2541667\cdot\Xi$ $\qquad \delta_3 := -0.4819444\cdot\Xi$ $\qquad t_3 := 419.5417$ $\qquad$ Dec. 1, 1996 6:00 pm

$T_1 := k\cdot(t_3 - t_2)$

$T_2 := k\cdot(t_3 - t_1)$

$T_3 := k\cdot(t_2 - t_1)$

These are the three normalized time differences associated with the three separate observations.

$T_1 = 0.68736835$ $\qquad T_2 = 1.51163445$ $\qquad T_3 = 0.82426609$

Now I need the geocentric Sun positions for the three times. These are from the 1996 Astronomical Almanac book, and I interpolate between two adjacent days in the Table to get the right Julian time.

$X_1 := -.963664$ $\qquad Y_1 := 0.271679$ $\qquad Z_1 := 0.117785$

$X_2 := -.86156452$ $\qquad Y_2 := -.456282$ $\qquad Z_2 := -.197827$

$X_3 := -0.33433726$ $\qquad Y_3 := -0.850871$ $\qquad Z_3 := -0.368908$

$R1 := \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} \qquad R2 := \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} \qquad R3 := \begin{bmatrix} X_3 \\ Y_3 \\ Z_3 \end{bmatrix}$ R is the solar position for each of the three times

$|R1| = 1.00813248$ $\qquad |R2| = 0.99479757$ $\qquad |R3| = 0.98582756$ $\qquad$ The Earth gets closer to the Sun in the Winter.

Next I calculate the geocentric unit vectors for each observation from the RA and Declinations measured. These are basically the angles to the comet from earth.... but I don't know the distance.

$\left(ux_i \;\; uy_i \;\; uz_i\right) := \left(\cos(\alpha_i)\cdot\cos(\delta_i) \;\; \sin(\alpha_i)\cdot\cos(\delta_i) \;\; \sin(\delta_i)\right)$

$$u1 := \begin{bmatrix} ux_1 \\ uy_1 \\ uz_1 \end{bmatrix} \qquad u2 := \begin{bmatrix} ux_2 \\ uy_2 \\ uz_2 \end{bmatrix} \qquad u3 := \begin{bmatrix} ux_3 \\ uy_3 \\ uz_3 \end{bmatrix}$$

$ux_2 = -0.10833159$
$uy_2 = -0.99183452$
$uz_2 = -0.06729457$

u1, u2, and u3 are the unit vectors (directions) from the earth to the comet for each of the three times, T1, T2, and T3

$|u1| = 1.00000000$ $\quad |u2| = 1.00000000$ $\quad |u3| = 1.00000000$ The vectors do have length 1.

Now Gauss determines the postions at time 2, by using information from times 1 and 3, and the Newtonian force law.

Since a simple 2-body orbit will be in one plane, then it is possible to describe the third position vector for the object as some linear combination of the other two positions, as long as the other two are not parallel to each other. So, for example r2= c1*r1 + c3*r3 What I will do is calculate the geometrical coefficients c1 and c3.

I calculate the geometrical coefficients from the relation of areas swept out per unit time and using Kepler's second Law. I am getting this from the book "Introduction to Celestial Mechanics", by S. W. McCuskey. First I need to define some coefficients that I will use later.

$$a_1 := \left(\frac{T_1}{T_2}\right) \qquad b_1 := \frac{1}{6}\cdot\left(\frac{T_1}{T_2}\right)\cdot\left[1-\left(\frac{T_1}{T_2}\right)^2\right]\cdot(T_2)^2 \qquad a_3 := \left(\frac{T_3}{T_2}\right) \qquad b_3 := \frac{1}{6}\cdot\left(\frac{T_3}{T_2}\right)\cdot\left[1-\left(\frac{T_3}{T_2}\right)^2\right]\cdot(T_2)^2$$

$a_1 = 0.45471863$ $\quad a_3 = 0.54528137$ $\quad b_1 = 0.13736773$ $\quad b_3 = 0.14591948$

$$A := \frac{\left[a_1\cdot(R1\cdot(u1\times u3))\right]-(R2\cdot(u1\times u3))+\left[a_3\cdot(R3\cdot(u1\times u3))\right]}{u1\cdot(u2\times u3)} \qquad A = 3.15407435$$

$$B := \frac{\left[b_1\cdot(R1\cdot(u1\times u3))\right]+\left[b_3\cdot(R3\cdot(u1\times u3))\right]}{(u1\cdot(u2\times u3))} \qquad B = -2.37224388$$

I need to solve two equations and two unknowns. They are nonlinear equations, so I could solve them graphically, or let the computer do the dirty work for me. These equations use the coefficients, which I have already calculated above.

$\rho 2 := 5$
$rr := 6$

Now I define starting (guesses) values for ρ and rr, the distances from the Earth and Sun to the comet, respectively, to be used later as guesses for the computer solution.

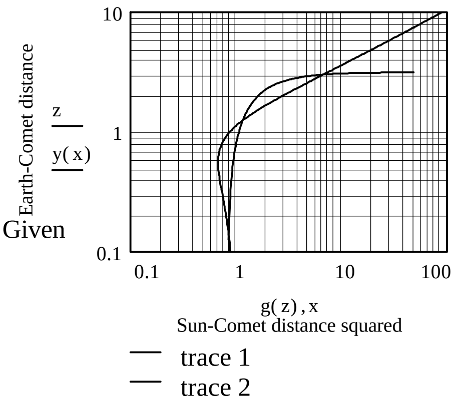
$x := 0.85, 0.860 .. 50$
$z := 0.1, 0.11 .. 10$

This left equation, below, came from first terms in an expansion of sector/triangle ratios by Gauss, and is approximate.

$$y(x) := A + \frac{B}{\left(x^{\frac{3}{2}}\right)}$$

$$g(z) := z^2 + (|R2|)^2 - (2\cdot z\cdot(u2\cdot R2))$$

The equation above is just the general form relating the three sides of a triangle, and is exact.



Given

I can solve the two equations graphically, by looking at the graph and seeing the points where the two curves intersect.

Only one intersection will be the real answer.There could be up to three different solutions, so I need to check several starting values when using the computer solver.

$$\left(A + \frac{B}{rr^3}\right) - \rho 2 = 0 \qquad (|\rho 2|)^2 + (|R2|)^2 - (2\cdot|\rho 2|\cdot(u2\cdot R2)) - rr^2 = 0$$

Or I use the computer routine "find" function to get the solutions.

$$Answer := find(rr, \rho 2) \qquad Answer = \begin{pmatrix} 2.59276927 \\ 3.01797134 \end{pmatrix}$$

$$rr := |Answer_0| \qquad \rho_2 := |(Answer)_1|$$

$rr = 2.59276927$ Sun-Comet distance at time 2

$\rho_2 = 3.01797134$ Earth-Comet distance at time 2

Now the coefficients that give position 2 as a linear combination of positions 1 and 3 can be calculated.

$$c_1 := a_1 + \frac{b_1}{(rr)^3} \qquad c_3 := a_3 + \frac{b_3}{(rr)^3}$$ EQ. (A).

$$\rho_1 := \frac{(R1\cdot(u2\times u3))\cdot c_1 - (R2\cdot(u2\times u3)) + c_3\cdot(R3\cdot(u2\times u3))}{c_1\cdot(u1\cdot(u2\times u3))}$$

$$\rho_3 := \frac{c_1\cdot(R1\cdot(u1\times u2)) - (R2\cdot(u1\times u2)) + c_3\cdot(R3\cdot(u1\times u2))}{c_3\cdot(u3\cdot(u1\times u2))}$$

So, finally I have solved for the position vectors of the comet at the three times (ρ1, ρ2, and ρ3). Now I can multiply the distances by their unit vectors from Earth and get the Sun-Comet heliocentric rectangular equatorial vector positions.

$$r10 := \rho_1\cdot u1 - R1 \qquad r20 := \rho_2\cdot u2 - R2 \qquad r30 := \rho_3\cdot u3 - R3$$

r is the heliocentric position vector of the comet

$$|r10| = 3.11297449 \qquad |r20| = 2.59276927 \qquad |r30| = 2.13471796$$

I'm going to correct my observation times for the speed of light now, and then iterate once on my coefficients, for an improved calculation.

$tc_1 := t_1 - 0.00577 \cdot \rho_1 \qquad tc_2 := t_2 - 0.00577 \cdot \rho_2 \qquad tc_3 := t_3 - 0.00577 \cdot \rho_3$

$T_1 := k \cdot \left(tc_3 - tc_2\right) \qquad T_2 := k \cdot \left(tc_3 - tc_1\right) \qquad T_3 := k \cdot \left(tc_2 - tc_1\right) \qquad n := \frac{T_3}{T_2} \qquad m := \frac{T_1}{T_2}$

$B_1 := \frac{1}{12} \cdot (m \cdot n - n + m) \cdot \left(T_2\right)^2 \qquad B_2 := \frac{1}{12} \cdot (m \cdot n + 1) \cdot \left(T_2\right)^2 \qquad B_3 := \frac{1}{12} \cdot ((m \cdot n - n) + m) \cdot \left(T_2\right)^2$

$c_3 := \frac{T_3}{T_2} \cdot \frac{1 + B_3 \cdot (|r30|)^{-3}}{1 - B_2 \cdot (|r20|)^{-3}} \qquad c_1 := \frac{T_1}{T_2} \cdot \frac{1 + B_1 \cdot (|r10|)^{-3}}{1 - B_2 \cdot (|r20|)^{-3}}$

These are the corrected coefficents, based on the first estimate of r1, r2, and r3. Now I can use the new c1 and c3, to get new and improved b1 and b3.

$b_1 := \left(c_1 - a_1\right) \cdot (|r20|)^3 \qquad b_3 := \left(c_3 - a_3\right) \cdot (|r20|)^3$

Now a new coefficient B can be calculated, and then new estimates for ρ2 and rr. The coefficient A has not changed, in this correction.

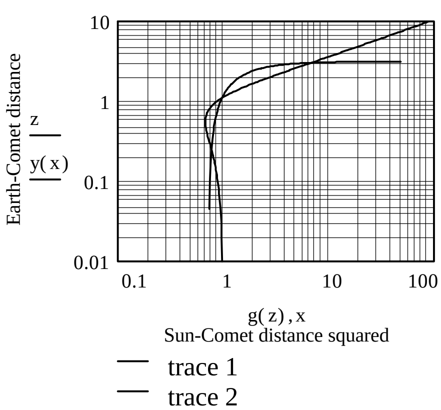
$B := \frac{\left[b_1 \cdot (R1 \cdot (u1 \times u3))\right] + \left[b_3 \cdot (R3 \cdot (u1 \times u3))\right]}{(u1 \cdot (u2 \times u3))} \qquad B = -1.97895091$

$\rho 2 := 5 \qquad x := 0.74, 0.75 .. 50$

$rr := 6 \qquad z := 0.01, 0.02 .. 10$

$g(z) := z^2 + (|R2|)^2 - (2 \cdot z \cdot (u2 \cdot R2))$

$y(x) := A + \frac{B}{\left(x^{\frac{3}{2}}\right)}$



The intersection points of these two curves give possible solution pairs of the Earth-Comet and Sun-Comet distances at Time 2.

Given

$$\left(A+\frac{B}{rr^3}\right)-\rho 2=0 \qquad (|\rho 2|)^2+(|R2|)^2-(2\cdot|\rho 2|\cdot(u2\cdot R2))-rr^2=0$$

$$\text{Answer} := \text{find}(rr,\rho 2) \qquad \text{Answer} = \begin{pmatrix} 2.61716294 \\ 3.04368104 \end{pmatrix}$$

$$rr := |\text{Answer}_0| \qquad \rho_2 := |(\text{Answer})_1|$$

$rr = 2.61716294$ Sun-Comet distance at time 2

$\rho_2 = 3.04368104$ Earth-Comet distance at time 2

$$c_1 := a_1 + \frac{b_1}{(rr)^3} \qquad c_3 := a_3 + \frac{b_3}{(rr)^3}$$

$$\rho_1 := \frac{(R1\cdot(u2\times u3))\cdot c_1-(R2\cdot(u2\times u3))+c_3\cdot(R3\cdot(u2\times u3))}{c_1\cdot(u1\cdot(u2\times u3))}$$

$$\rho_3 := \frac{c_1\cdot(R1\cdot(u1\times u2))-(R2\cdot(u1\times u2))+c_3\cdot(R3\cdot(u1\times u2))}{c_3\cdot(u3\cdot(u1\times u2))}$$

$$r1 := \rho_1\cdot u1 - R1 \qquad r2 := \rho_2\cdot u2 - R2 \qquad r3 := \rho_3\cdot u3 - R3$$

$$|r1| = 3.15841934 \qquad |r2| = 2.61716294 \qquad |r3| = 2.14587940$$

Now I have to go to the ecliptic plane. Since the Earth axis is tilted by 23 degrees, 27 minutes to the plane of the ecliptic this means I have to do a rotation.

$$\varepsilon := 23.4396\cdot\left(\frac{2\cdot\pi}{360}\right) \qquad N := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\varepsilon) & \sin(\varepsilon) \\ 0 & -\sin(\varepsilon) & \cos(\varepsilon) \end{bmatrix}$$

N is the 2D rotation matrix through an angle ε radians

Since I am running out of r's to use, I will choose S to denote the new heliocentric rectangular ecliptic coordinates, at each of the three times.

$$S1 := N\cdot r1$$

$$S2 := N\cdot r2$$

$$S3 := N\cdot r3 \qquad S1 = \begin{pmatrix} 0.67412158 \\ -2.97411073 \\ 0.82209379 \end{pmatrix}$$

$$|S1| = 3.15841934$$

$$|S2| = 2.61716294$$

$$|S3| = 2.14587940$$

Just checking that the coordinate rotation did not change the distances.

SO, FROM THE KNOWN POSITIONS AT THE THREE TIMES, THE ORBIT PARAMETERS CAN BE CALCULATED NEXT.

Finally, I can calculate the six standard orbital elements, in the heliocentric ecliptic coordinate. system.

$$e3 := \frac{S1 \times S3}{|S1 \times S3|} \qquad e3 = \begin{pmatrix} -0.97497798 \\ -0.22225449 \\ -0.00456855 \end{pmatrix}$$

e3 is the new unit vector, which is perpendicular to the plane of the orbit, ... that is, perpendicular to the two position measurements, vectors S1 and S3.

$\iota := \mathrm{acos}\left(e3_2\right) \qquad \iota = 1.57536489$ iota (ι) is the inclination of the orbit with respect to the ecliptic

$$\Omega := \mathrm{atan}\left(-\frac{e3_0}{e3_1}\right) \qquad \Omega = -1.34666776 \qquad \Omega := \Omega + 2\cdot\pi$$

(Add 2π if numerator and denominator negative)

Omega (Ω) is the longitude of the ascending node of the comet with respect to the line of the vernal equinox. The other two unit vectors in the orbital plane are now defined as:

$e2_0 := -\sin(\Omega)\cdot\cos(\iota) \qquad e2_1 := \cos(\Omega)\cdot\cos(\iota) \qquad e2_2 := \sin(\iota)$

$e1_0 := \cos(\Omega) \qquad e1_1 := \sin(\Omega) \qquad e1_2 := 0$

Next I will solve for two equations, in two unknowns, $e\cos\omega$ and $e\sin\omega$, which I refer to as XX and YY e is the eccentricity, and ω is called the argument of the perihelion.

$XX := 0.1 \quad YY := 0.1$ Starting guess.

Given

$$XX\cdot((S1 - S2)\cdot e1) + YY\cdot((S1 - S2)\cdot e2) - (|S2| - |S1|) = 0$$

$$XX\cdot((S1 - S3)\cdot e1) + YY\cdot((S1 - S3)\cdot e2) - (|S3| - |S1|) = 0$$

$$SolveTwo := \mathrm{find}(XX, YY) \qquad SolveTwo = \begin{pmatrix} -0.62431913 \\ 0.75445748 \end{pmatrix}$$

$XX := SolveTwo_0 \qquad YY := SolveTwo_1$

$ecc := \sqrt{XX^2 + YY^2} \qquad ecc = 0.97927548$

If eccentricity (ecc) is > 1, the orbit is a hyperbola..... and does not return.

$$\omega := \mathrm{atan}\left(\frac{YY}{XX}\right) \qquad \omega = -0.87950618$$

$\omega := \omega + \pi$

If XX is negative, and YY is positive, add π to the arctan to put the angle in the proper quadrant.

$$a := \frac{|S2| + XX\cdot(S2\cdot e1) + YY\cdot(S2\cdot e2)}{\left(1 - ecc^2\right)} \qquad a = 45.70426141$$

(IF a is NEGATIVE, I have a problem !)

$$P := \frac{2\cdot\pi}{k\cdot\sqrt{M0}}\cdot a^{\frac{3}{2}}$$

Kepler's Third Law...... relating period P to the semimajor axis raised to the 3/2 power.

$P = 1.12858291\cdot 10^{5}$ $\quad P\_years := \frac{P}{365.25}$ $\quad P\_years = 308.99$

$E := -\operatorname{acos}\left(\frac{a - |S2|}{a\cdot ecc}\right)$ $\quad E = -0.27402958$ E is the "eccentric anomaly" angle at time t2

$M := E - ecc\cdot\sin(E)$ $\quad M = -0.00902505$ M is the "mean anomaly" angle at time t2

$T := t_{2} - \frac{M\cdot P}{2\cdot\pi}$ $\quad T = 541.69083404$ T is the time of perihelion passage, in my Julian date units.

$q := a\cdot(1 - ecc)$ $\quad I := \frac{360}{2\cdot\pi}\cdot\iota$ $\quad O := \frac{360}{2\cdot\pi}\cdot\Omega$ $\quad W := \frac{360}{2\cdot\pi}\cdot\omega$

$P\_years = 308.99$ Orbital Period (in years)

$a = 45.704$ Semimajor axis (in AU)

P= 2364
a= 186.85

The calculated orbital elements from my data and calculations are:

For comparison, here is JPL Solution #48

$ecc = 0.97927548$	eccentricity			.995107808
$q = 0.94719896$	distance of perihelion closest approach to sun (in AU.)			.914103842
$\iota = 1.57536489$	inclination angle from ecliptic	$I = 90.26$	degrees	89.429449
$\omega = 2.26208647$	Argument of perihelion	$W = 129.61$	degrees	130.5910916
$\Omega = 4.93651755$	longitude of ascending node	$O = 282.84$	degrees	282.470692
$T = 541.69083404$	Time of perihelion passage			539.6353

My eccentricity is too small, but at least it's under 1. My distance of perihelion closest approach to the sun is 3% too big. However, my inclination, argument of the perihelion, and longitude of the ascending node are very close. My time of perihelion is also only two days late. Overall, it is really a quite accurate set of orbital parameters. The main problem is the eccentricity, and the orbital period, which is much shorter than the real 2300 years.