Fundamental Plasma Physics Equations

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Outline

- Plasma
- Maxwell-Lorentz description of plasma
- Kinetic description of plasma
  - distribution function
  - kinetic equation
  - collision operator
- Multi-fluid description of plasma
  - hierarchy of fluid equations
  - typical truncation $\Rightarrow$ continuity, momentum, and energy conservation
  - heat fluxes and viscosities
  - Braginskii closure for collisional plasma
- Single-fluid description of plasma (**magnetohydrodynamics**)

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Plasma

- Plasma is a **quasineutral** gas of charged and neutral particles in which **collective effects** dominate over collisions
  - **quasineutrality** $\Rightarrow$ the external potential shielding length, Debye length λ_D, is much smaller than the plasma dimensions, L:

$$\lambda_D \equiv \sqrt{\frac{T_e}{4\pi e^2 n_e}} \ll L$$

  - can only talk about Debye shielding if the number of particles in the Debye sphere ($\equiv$ **plasma parameter**), N_D, is large:

$$N_D \equiv \frac{4}{3}\pi\lambda_D^3 n_e \ggg 1$$

  - **collective behavior** $\Rightarrow$ the frequency of typical plasma oscillations, ω, is large compared with collision frequency with neutral atoms, τ^{-1}:

$$\omega\tau > 1$$

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Maxwell-Lorentz Description of Plasma

Euler-Lagrange Equations

- View plasma as a collection of charged particles (use j to denote plasma species)
- Each (non-relativistic) charged particle can be described by the **Lagrangian**:

$$L_j(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) = \frac{m_j |\dot{\boldsymbol{x}}|^2}{2} - e_j \Phi(\boldsymbol{x}, t) + \frac{e_j}{c} \dot{\boldsymbol{x}} \cdot \boldsymbol{A}(\boldsymbol{x}, t)$$

- The Euler-Lagrange equations of motion,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L_j}{\partial \dot{\boldsymbol{x}}} \right) = \frac{\partial L_j}{\partial \boldsymbol{x}}$$

are equivalent to the **Lorentz force law**:

$$m_j \ddot{\boldsymbol{x}} = e_j \boldsymbol{E} + \frac{e_j}{c} \dot{\boldsymbol{x}} \times \boldsymbol{B},$$

with

$$\boldsymbol{E} \equiv -\boldsymbol{\nabla} \Phi - \frac{1}{c} \frac{\partial \boldsymbol{A}}{\partial t}, \qquad \boldsymbol{B} \equiv \boldsymbol{\nabla} \times \boldsymbol{A}$$

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Maxwell Equations

- $\boldsymbol{E}$ and $\boldsymbol{B}$ obey the Maxwell equations:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = 4\pi\rho_c, \qquad \boldsymbol{\nabla}\times\boldsymbol{B} - \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t} = \frac{4\pi}{c}\boldsymbol{J}, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \qquad \boldsymbol{\nabla}\times\boldsymbol{E} + \frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t} = 0$$

- The interactions of the fields and the particles occur only through the plasma charge density ρ_c and current $\boldsymbol{J}$

- If we assume $i = 1, \ldots, N_j$ particles of the species j at positions $(\boldsymbol{x}_j^i, \boldsymbol{v}_j^i)$ then

$$\rho_c(\boldsymbol{x},t) = \sum_j e_j n_j(\boldsymbol{x},t) \quad \text{with} \quad n_j(\boldsymbol{x},t) = \sum_{i=1}^{N_j} \delta(\boldsymbol{x} - \boldsymbol{x}_j^i(t)),$$

$$\boldsymbol{J}(\boldsymbol{x},t) = \sum_j e_j n_j(\boldsymbol{x},t)\boldsymbol{V}_j(\boldsymbol{x},t) \quad \text{with} \quad n_j(\boldsymbol{x},t)\boldsymbol{V}_j(\boldsymbol{x},t) = \sum_{i=1}^{N_j} \boldsymbol{v}_j^i(t)\delta(\boldsymbol{x}-\boldsymbol{x}_j^i(t))$$

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Maxwell-Lorentz Description in not Useful

- The Lorentz force law + the Maxwell equations give exact and complete description of plasma motions
- These equations are **not useful** since it is neither possible, nor even desirable, to know positions and velocities of all plasma particles
- Plasma physics derives approximate sets of equations that are tractable
- **Kinetic theory** replaces the Lorentz force law with an equation for smoothed particle distribution function in phase space
- **Fluid theory** derives equations directly for plasma density and current and usually requires only minimal information about the particle distribution

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Kinetic Description of Plasma

Distribution Function

- Kinetic theory provides, for each species j, a **distribution function** $f_j(\boldsymbol{x}, \boldsymbol{v}, t)$, such that $f_j(\boldsymbol{x}, \boldsymbol{v}, t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}v_x\,\mathrm{d}v_y\,\mathrm{d}v_z$ is the number of species-j particles around point (x, y, z), having velocity (v_x, v_y, v_z), at time t

- It is sufficient to determine f_j as a function of $\boldsymbol{E}$ and $\boldsymbol{B}$ to close the Maxwell equations, since

$$\rho_c(\boldsymbol{x}, t) = \sum_j e_j \int \mathrm{d}^3v\, f_j(\boldsymbol{x}, \boldsymbol{v}, t)$$

$$\boldsymbol{J}(\boldsymbol{x}, t) = \sum_j e_j \int \mathrm{d}^3v\, \boldsymbol{v} f_j(\boldsymbol{x}, \boldsymbol{v}, t)$$

- The distribution function f_j is obtained from a **kinetic equation**

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Kinetic Equation

- Assume no atomic and nuclear reactions $\Rightarrow$ particles of each species are conserved

- Then each distribution function is conserved along particle trajectories:

$$\frac{\partial f_j}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} f_j + \boldsymbol{a}_j \cdot \frac{\partial f_j}{\partial \boldsymbol{v}} = 0, \qquad \boldsymbol{a}_j = \frac{e_j}{m_j}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}\right)$$

- $\boldsymbol{E}$ and $\boldsymbol{B}$ in $\boldsymbol{a}_j$ include external forces and electromagnetic interactions of all the particles $\Rightarrow$ $\boldsymbol{a}_j$ is pathologically jagged (depends on the positions and velocities of all the particles)

- Average f_j and $\boldsymbol{a}_j$ over an ensemble of macroscopically equivalent plasmas $\Rightarrow$ get smooth functions $\bar{f}_j \equiv \langle f_j \rangle_{\text{ensemble}}$, $\bar{\boldsymbol{a}}_j \equiv \langle \boldsymbol{a}_j \rangle_{\text{ensemble}}$

- Because of particle interactions $\langle \boldsymbol{a}_j \cdot (\partial f_j / \partial \boldsymbol{v}) \rangle_{\text{ensemble}} \neq \bar{\boldsymbol{a}}_j \cdot (\partial \bar{f}_j / \partial \boldsymbol{v}) \Rightarrow$ must include collisions

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Collision Operator

- As a result, all kinetic equations have the form (suppressing the overbars):

$$\frac{\partial f_j}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} f_j + \boldsymbol{a}_j \cdot \frac{\partial f_j}{\partial \boldsymbol{v}} = C_j(f_j)$$

- Collision operator C_j provides a statistical account of random interactions of particle species j with other particles; **binary** collisions are normally considered
- There is **no exact** version of the collision operator, only approximations
- Examples are **Boltzmann** collision operator, **Fokker-Planck** collision operator
- Collision operator $C_j = \sum_k C_{jk}(f_j, f_k)$ is
  - bilinear, that is $C_{jk}(f_j, f_k)$ are linear in f_j, f_k
  - cause entropy to increase
  - Galilean invariant, that is $C_{jk}(f_j(\boldsymbol{v} - \boldsymbol{V}), f_k(\boldsymbol{v} - \boldsymbol{V})) = C_{jk}(f_j(\boldsymbol{v}), f_k(\boldsymbol{v}))$
  - conserve particles, momentum, and energy at each $\boldsymbol{x}$

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Kinetic Description of Plasma in Rarely Effective

- Evaluation of a 6D distribution function is difficult: analytical solutions to a kinetic equation are rare and numerical are expensive

- Closing the Maxwell equations with the kinetic equation is inefficient since only two lowest order moments of f_j (namely density and flow) are required, while f_j contains enormously more information

- In other words, f_j itself does not matter, since closure of the Maxwell equations involves only an equivalence class of distributions, whose different members, however various, have the same density and flow

- Fluid descriptions can be much more effective

- However, sometimes one cannot avoid solving a kinetic equation $\Rightarrow$ use a somehow simplified version (e.g. **drift kinetics, gyrokinetics**)

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Multi-fluid Description of Plasma

Continuity, Momentum, and Energy Conservation

- Evaluate $\int \mathrm{d}^3 v$, $\int \mathrm{d}^3 v\, \boldsymbol{v}$, and $\int \mathrm{d}^3 v\, (m_j v^2/2)$ integrals of a kinetic equation

- Resulting are **continuity**, **momentum conservation**, and **energy conservation** equations for each particle species j:

$$\frac{\partial n_j}{\partial t} + \boldsymbol{\nabla} \cdot (n_j \boldsymbol{V}_j) = 0,$$

$$m_j n_j \left(\frac{\partial \boldsymbol{V}_j}{\partial t} + \boldsymbol{V}_j \cdot \boldsymbol{\nabla} \boldsymbol{V}_j \right) + \boldsymbol{\nabla} p_j + \boldsymbol{\nabla} \cdot \overleftrightarrow{\pi}_j = e_j n_j \left(\boldsymbol{E} + \frac{1}{c} \boldsymbol{V}_j \times \boldsymbol{B} \right) + \boldsymbol{F}_j,$$

$$\frac{\partial}{\partial t} \left(\frac{3}{2} p_j + \frac{1}{2} m_j n_j V_j^2 \right) + \boldsymbol{\nabla} \cdot \left[\left(\frac{5}{3} p_j + \frac{1}{2} m_j n_j V_j^2 \right) \boldsymbol{V}_j + \boldsymbol{q}_j + \overleftrightarrow{\pi}_j \cdot \boldsymbol{V}_j \right] = W_j + \boldsymbol{V}_j \cdot (\boldsymbol{F}_j + e_j n_j \boldsymbol{E})$$

for **density** $n_j \equiv \int \mathrm{d}^3 v f_j$, **flow velocity** $\boldsymbol{V}_j \equiv n_j^{-1} \int \mathrm{d}^3 v \boldsymbol{v} f_j$, and **pressure** $p_j \equiv \int \mathrm{d}^3 v (m_j w^2/3) f_j$ with $\boldsymbol{w} \equiv \boldsymbol{v} - \boldsymbol{V}_j$

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Hierarchy of Fluid Equations

- An equation for every moment of f_j contains higher order moments of f_j and possibly some moments of the collision operator
  - continuity for $n_j = \int \mathrm{d}^3v\, f_j$ contains $\boldsymbol{V}_j = n_j^{-1} \int \mathrm{d}^3v\, \boldsymbol{v} f_j$
  - momentum for $\boldsymbol{V}_j = n_j^{-1} \int \mathrm{d}^3v\, \boldsymbol{v} f_j$ contains $p_j = \int \mathrm{d}^3v\, (m_j w^2/3) f_j$, **viscosity tensor** $\overleftrightarrow{\pi}_j = \int \mathrm{d}^3v\, m_j [\boldsymbol{w}\boldsymbol{w} - (w^2/3)\, \overleftrightarrow{I}\,] f_j$ with $\overleftrightarrow{I}$ = unit tensor, and **friction force** $\boldsymbol{F}_j \equiv \int \mathrm{d}^3v\, m_j \boldsymbol{v} C_j$
  - energy for $(3/2)p_j + (1/2)m_j n_j V_j^2 = \int \mathrm{d}^3v\, (m_j v^2/2) f_j$ contains **heat flux** $\boldsymbol{q}_j = \int \mathrm{d}^3v\, (m_j/2) w^2 \boldsymbol{w} f_j$ and energy exchange term $W_j \equiv \int \mathrm{d}^3v\, (m_j w^2/2) C_j$
- Can evaluate $\int \mathrm{d}^3v\, (m_j v^2 \boldsymbol{v}/2)$ integral of a kinetic equation to obtain an equation for $\boldsymbol{q}_j$ and keep on going $\Rightarrow$ an infinite hierarchy of fluid equations that is equivalent to the full kinetic equation
- Alternatively, can use a kinetic equation to evaluate $\overleftrightarrow{\pi}_j$, $\boldsymbol{q}_j$, $\boldsymbol{F}_j$, and W_j in terms of n_j, $\boldsymbol{V}_j$, and p_j $\Rightarrow$ **multi-fluid closures**

Multi-fluid Closures

- To evaluate $\overleftrightarrow{\pi}_j$, $\boldsymbol{q}_j$, $\boldsymbol{F}_j$, and W_j in terms of n_j, $\boldsymbol{V}_j$, and p_j have to solve a kinetic equation for f_j analytically in terms of n_j, $\boldsymbol{V}_j$, and p_j and then evaluate its appropriate velocity integrals

- This has only been done **rigorously** so far only for **collisional plasmas** when

$$\frac{\lambda_j}{L} \ll 1,$$

  where $\lambda_j \sim v_{Tj}\tau_j$ is the species j mean-free path with v_{Tj} the thermal speed and τ_j the characteristic collision time, and L is the characteristic plasma length scale

- The best-known two-fluid closure (for plasma consisting of electrons and one ion species) was obtained by the Soviet plasma physicist **S. I. Braginskii** in 1952

- Instead of looking at the exact expressions let's make some estimates

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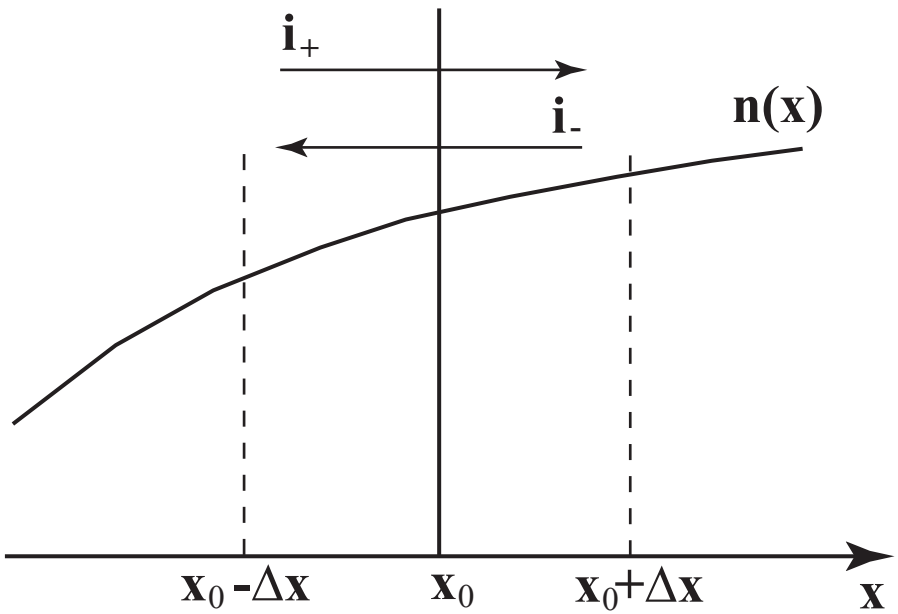
Gas-kinetic Estimates



- Consider diffusion of particles with a density profile $n(x) \neq \text{constant}$
- Assume that during the time τ between collisions every particle can move a distance Δx to the left or to the right with a probability $1/2$
- In unit time $1/2$ of all particles experiencing collisions in a layer $(x_0 - \Delta x, x_0)$ would cross the plane $x = x_0$ from left to right

- Expanding $n(x) = n(x_0) + (\partial n/\partial x)|_{x=x_0}(x - x_0)$ we find the flux from left to right:

$$i_+ = \frac{1}{2}\int_{x_0-\Delta x}^{x_0} \frac{n(x)}{\tau}\mathrm{d}x = \frac{1}{2}\left[n(x_0) - \frac{\partial n}{\partial x}\frac{\Delta x}{2}\right]\frac{\Delta x}{\tau}$$

Gas-kinetic Estimates (Continued)

- Similarly, the particle flux from right to left is

$$i_- = \frac{1}{2}\left[n(x_0) + \frac{\partial n}{\partial x}\frac{\Delta x}{2}\right]\frac{\Delta x}{\tau}$$

- Then, the diffusion flux along x, $i_x = i_+ - i_- = -D\dfrac{\partial n}{\partial x}, \qquad D \equiv \dfrac{(\Delta x)^2}{2\tau}$

- Similarly, a temperature gradient along x produces a heat flux

$$q_x = -\kappa\frac{\partial T}{\partial x}, \qquad \kappa \sim \frac{n(\Delta x)^2}{\tau} \sim nD$$

- Similarly, if a velocity V_y changes along x then there is a flux π_{yx} of the y-directed momentum along x:

$$\pi_{yx} = -\eta\frac{\partial V_y}{\partial x}, \qquad \eta \sim \frac{mn(\Delta x)^2}{\tau} \sim mnD$$

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Parallel and Perpendicular Heat Fluxes

- The expression for κ provides estimates for electron and ion heat fluxes along and across magnetic field $\boldsymbol{B}$

- Heat conduction along $\boldsymbol{B}$
  - $\Delta x_j \sim \lambda_j (\equiv \text{mean free path}) \sim v_{Tj}\tau_j \Rightarrow \kappa_{\|j} \sim n_j \lambda_j^2/\tau_j \sim n_j T_j \tau_j / m_j$

- Heat conduction across $\boldsymbol{B}$
  - $\Delta x_j \sim r_j (\equiv \text{gyroradius}) \sim v_{Tj}/\Omega_j \Rightarrow \kappa_{\perp j} \sim n_j r_j^2/\tau_j \sim n_j T_j/(m_j \Omega_j^2 \tau_j)$

- Then, $\kappa_{\|j}/\kappa_{\perp j} \sim (\Omega_j \tau_j)^2$

- For Coulomb collisions with $T_e \sim T_i$, $\tau_e/\tau_i \sim (m_e/m_i)^{1/2} \lesssim 1/40 \ll 1$
  - $\kappa_{\|e}/\kappa_{\|i} \sim (m_i/m_e)^{1/2} \gg 1 \Rightarrow$ electron heat conduction along $\boldsymbol{B}$ is higher
  - $\kappa_{\perp e}/\kappa_{\perp i} \sim (m_e/m_i)^{1/2} \ll 1 \Rightarrow$ ion heat conduction across $\boldsymbol{B}$ is higher

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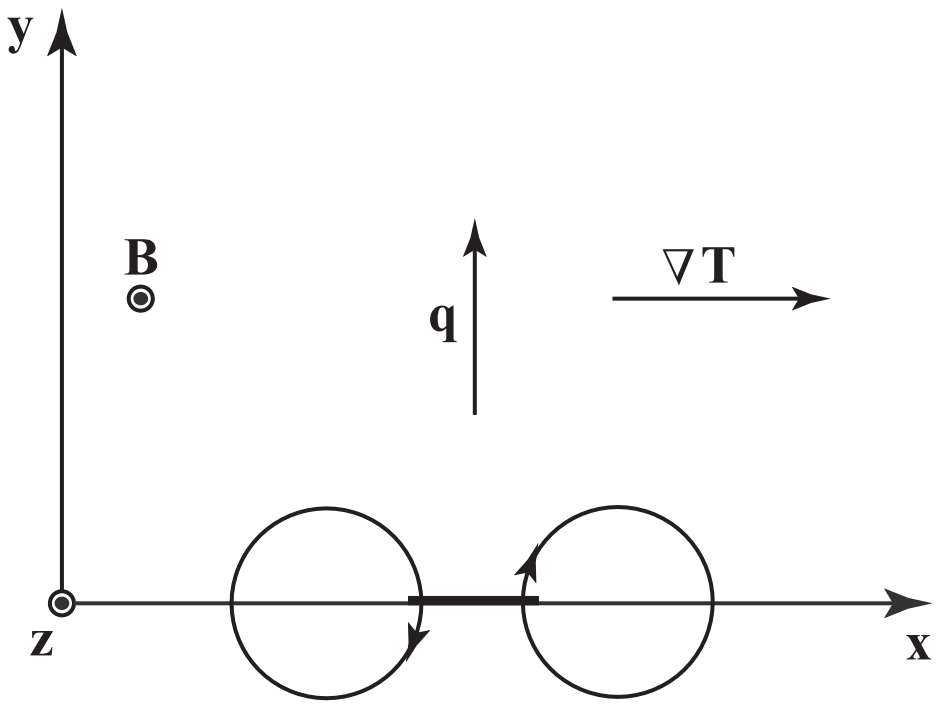
Diamagnetic Heat Fluxes



- In addition to the parallel and perpendicular heat fluxes just considered, which are parallel to ∇T, there exist electron and ion heat fluxes **perpendicular to** ∇T
- Occur since particles gyrating up (see figure) are hotter (and therefore faster) than those gyrating down
- If the up and down particle fluxes, $\Gamma_\pm \sim n_j v_{Tj}$, are equal, fractions $(r_j/T_j)(\partial T_j/\partial x)$ of the up and down heat fluxes, $q_\pm \sim n_j T_j v_{Tj}$, remain uncompensated

- This leads to $q_y \sim (r_j/T_j)(\partial T_j/\partial x) n_j T_j v_{Tj} \sim (c n_j T_j/eB)(\partial T_j/\partial x)$
- These **diamagnetic** heat fluxes carry heat along isotherms and have opposite signs for e and i

Braginskii's Expressions for $\boldsymbol{q}_j$ etc.

- In his work Braginskii obtained for the ion heat flux using kinetic theory

$$\boldsymbol{q}_i = -\underbrace{\left(3.9\frac{n_i T_i \tau_i}{m_i}\right)}_{\kappa_{\parallel i}} \boldsymbol{\nabla}_{\parallel} T_i - \underbrace{\left(2\frac{n_i T_i}{m_i \Omega_i^2 \tau_i}\right)}_{\kappa_{\perp i}} \boldsymbol{\nabla}_{\perp} T_i + \left(\frac{5}{2}\frac{c n_i T_i}{e_i B}\right) \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} T_i$$

  with $\hat{\boldsymbol{b}} \equiv \boldsymbol{B}/B$

- Electron heat flux contains similar temperature gradient terms **plus some additional terms**

- Expressions for viscous stress tensors $\overleftrightarrow{\pi}_j$, electron-ion friction forces $\boldsymbol{F}_j$, and electron-ion energy exchange terms W_j also can be estimated using the gas-kinetic-like analysis and were obtained exactly by Braginskii from kinetic theory

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Single-fluid Description of Plasma

Motivation for Single-fluid Description

- Exact multi-fluid equations require kinetic evaluation of $\overleftrightarrow{\pi}_j$, $\boldsymbol{q}_j$, $\boldsymbol{F}_j$, and W_j
- People have not figured out yet how to do this for hot fusion-grade plasmas or low-density astrophysical plasmas, which have very long mean-free paths compared with characteristic plasma length scales
- Rigorously speaking, have to use full Maxwell-kinetic formalism
- However, certain important phenomena can be understood by using very simple approximations for these quantities or even by neglecting some of them
- Magnetohydrodynamics (**MHD**) does just this by writing fluid equations for a single fluid - highly conductive magnetized plasma

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Assumptions for MHD Equations

- Assume plasma consists of electrons and singly-charged ions
- Concentrate on low-frequency, long-wavelength phenomena $\Rightarrow$ **quasineutrality** $n_e \approx n_i \equiv n$
- Assume the ion gyroradius is small, $r_i \ll L$ (strong $\boldsymbol{B}$), and particle mean-free paths are small, $\lambda_i \sim \lambda_e \ll L$ (strong collisionality)
- Neglect heat fluxes, viscosities, electron-ion friction forces (in Ideal MHD), and electron-ion energy exchange terms (equivalent to assuming $T_e = T_i$)
- Neglect electron mass, $m_e = 0$
- Introduce mass density $\rho \equiv m_i n$, fluid velocity $\boldsymbol{V} = \boldsymbol{V}_i$, and total pressure $p \equiv p_i + p_e$
- As a result get "very simple" MHD fluid equations

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Ideal MHD Equations

- Rewrite two-fluid equations as (with $\mathrm{d}/\mathrm{d}t \equiv \partial/\partial t + \boldsymbol{V}\cdot\boldsymbol{\nabla}$)

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\,\boldsymbol{\nabla}\cdot\boldsymbol{V} = 0$$

$$\rho\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \frac{1}{c}\boldsymbol{J}\times\boldsymbol{B} - \boldsymbol{\nabla}p$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{5}{3}p\,\boldsymbol{\nabla}\cdot\boldsymbol{V} = 0$$

$$\boldsymbol{E} + \frac{1}{c}\boldsymbol{V}\times\boldsymbol{B} = 0$$

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t}$$

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$$

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Comments on MHD Equations

- All MHD assumptions but the strong collisionality assumption, $\lambda_i \sim \lambda_e \ll L$, are satisfied for fusion-grade plasmas

- Does it mean that we should not use MHD?

- No! There are deep underlying reasons why MHD is actually valid well beyond its formal applicability region and is thereby useful but do not have time to go into this

- The rule of thumb:
  - if Ideal MHD predicts instability you are dead
  - if Ideal MHD predicts stability some other unpleasant instabilities can still exist but you might have a chance

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Conclusions

- We considered four ways of describing plasma: Lagrange equations, kinetic equations, multi-fluid equations, single-fluid equations AND Maxwell equations
- Lagrange (or Lorentz) equations are exact but not useful since need positions and velocities of all particles
- Kinetic equations for particle distribution functions are more tractable but give more information then required and still are very complicated
- Multi-fluid equations are much simpler, describe physically observable quantities (such as densities, flow velocities, and temperatures or pressures), but can only be rigorously derived for collisional plasmas
- Single-fluid MHD equations are simplest, work well beyond their formal applicability region, and provide a useful simplified plasma description

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