Force-free magnetic relaxation in a driven compact toroid

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Overview

- Force-free: $\mathbf{j} = \sigma \mathbf{B}$ and $\mathbf{B} \cdot \nabla \sigma = 0$.
- Taylor state: σ is a global constant.
- Flux amplification in driven compact toroids depends on the Jensen-Chu resonance.
- In Taylor's theory, accessible magnetic configuration bounded by the first Jensen-Chu resonance.
- Force-free but partially relaxed state (nonlinearity) regularizes Jensen-Chu singularity.
- Additional accessible magnetic configurations.

Force-free relaxed state

- Why force-free?

$$\nabla \times \mathbf{B} = \sigma \mathbf{B}, \;\; \mathbf{B} \cdot \nabla \sigma = 0$$

  - Transport is poor, so ∇p is small.
  - Small electric field, so magnetic Mach number is small.

- Taylor state: σ global constant.

  - $\mathbf{B} \cdot \mathbf{n}|_{\partial\Omega} = 0$: eigenvalue problem
  - $\mathbf{B} \cdot \mathbf{n}|_{\partial\Omega} \neq 0$: linear perturbation problem. (singularity at resonance)

- Partially relaxed state: σ flux function.

  - $\mathbf{B} \cdot \mathbf{n}|_{\partial\Omega} = 0$: nonlinear problem (multiple solution branch).
  - $\mathbf{B} \cdot \mathbf{n}|_{\partial\Omega} \neq 0$: nonlinear perturbation problem. (singularity regularized)
  - Contrary to Kitson-Browning result (1990).

Jensen-Chu formulation

With the expansion (Jensen and Chu, 1984)

$$\begin{aligned}
\mathbf{A} &= \mathbf{A}_I + \sum_\nu \alpha_\nu \mathbf{a}_\nu. \\
\nabla \times \nabla \times \mathbf{A}_I &= 0, \;\; \nabla \times \mathbf{A}_I \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{B} \cdot \mathbf{n}|_{\partial\Omega}. \\
\nabla \times \nabla \times \mathbf{a} &= \lambda_\nu \nabla \times \mathbf{a}_\nu, \;\; \mathbf{a}_\nu|_{\partial\Omega} = 0.
\end{aligned}$$

Jensen-Chu solution

$$\begin{aligned}
\alpha_\nu &= \sigma \frac{\lambda_\nu/|\lambda_\nu|}{\sigma - \lambda_\nu} \int \mathbf{a}_\nu \cdot \nabla \times \mathbf{A}_I dV = \sigma \frac{\lambda_\nu/|\lambda_\nu|}{\sigma - \lambda_\nu} I_\nu, \\
I_\nu &\equiv \int \mathbf{a}_\nu \cdot \nabla \times \mathbf{A}_I dV. \\
K &= \int \mathbf{A} \cdot \nabla \times \mathbf{A} dV = \sum_\nu I_\nu^2 \frac{\lambda_\nu}{|\lambda_\nu|} [1 - \frac{\lambda_\nu^2}{(\sigma - \lambda_\nu)^2}] + K_V. \\
E &= \frac{1}{2} \int (\nabla \times \mathbf{A}_I)^2 dV = \frac{1}{2} \sum_\nu I_\nu^2 |\lambda_\nu| \frac{\sigma^2}{(\sigma - \lambda_\nu)^2} + E_V.
\end{aligned}$$

Resonant: $I_\nu \neq 0$; Non-Resonant: $I_\nu = 0$.

Axisymmetric, resonant case

Force-free Grad-Shafranov eq.

$$\mathbf{B} = G(\chi)\nabla\varphi + \nabla\varphi \times \nabla\chi.$$

$$y\Delta^*\chi + GdG/d\chi = \Delta^*\chi + \sigma^2\chi = 0.$$

Jensen-Chu decomposition

$$\chi = \chi_0 + \sum_i \alpha_i \chi_i.$$

Vacuum field

$$\Delta^*\chi_0 = 0, \;\; \chi_0|_{\partial\Omega} = \chi|_{\partial\Omega}.$$

Expansion bases

$$\Delta^*\chi_i + \sigma_i^2\chi_i = 0, \;\; \chi_i|_{\partial\Omega} = 0.$$

Expansion coefficients and Jensen-Chu singularities

$$\alpha_i = \frac{\sigma^2}{\sigma_i^2 - \sigma^2}\langle\chi_0\chi_i\rangle$$

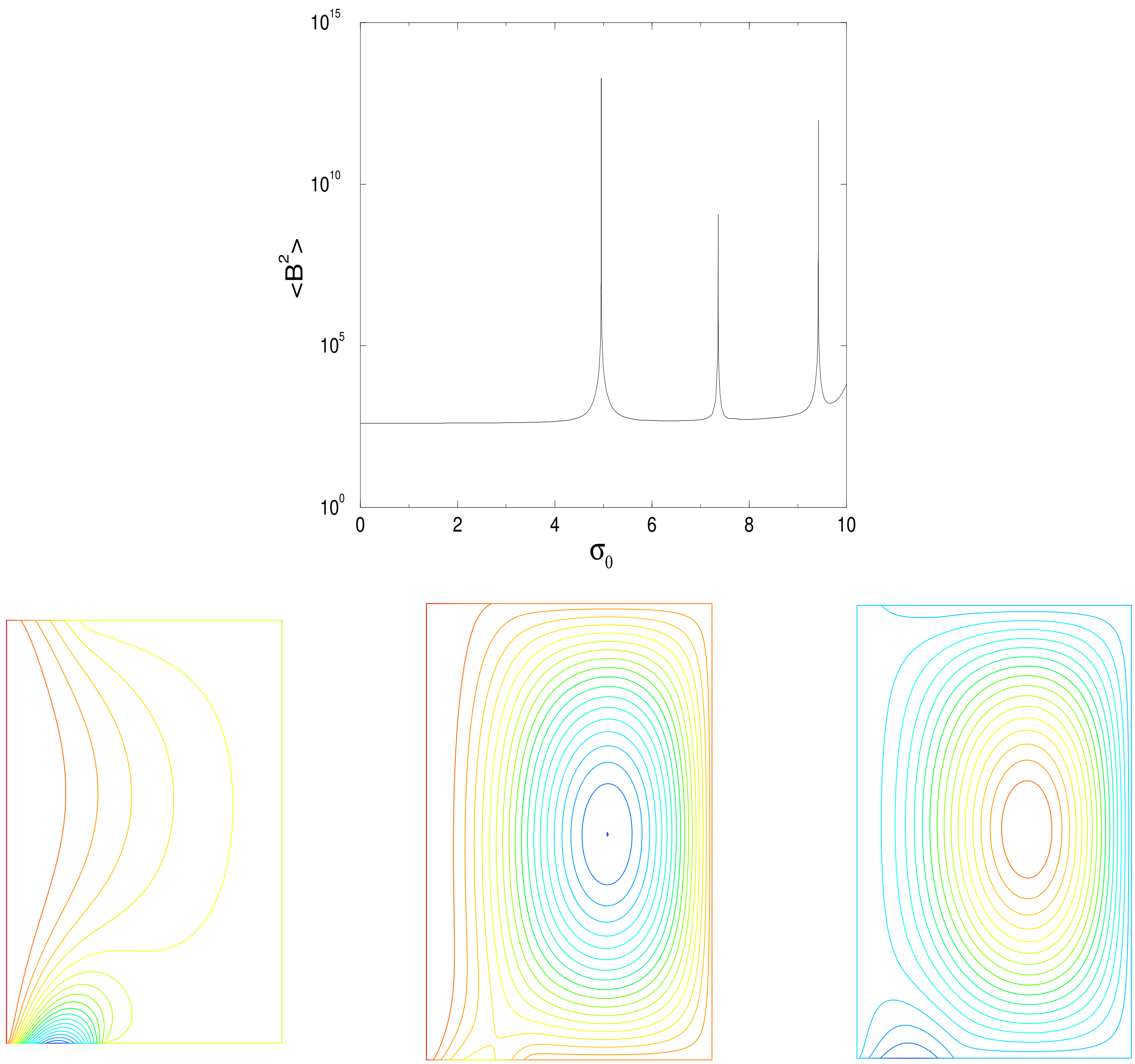

Figure 1: Top: Magnetic energy versus σ_0. Bottom: flux contours, left: vacuum field, middle: $\sigma_0 = 4.75$ (flux amplification), right: $\sigma_0 = 5.15$ (flipped spheromak)

Force-free but partially relaxed

Slight deviation from Taylor state

$$\sigma(\chi) = \sigma_0(1 + \epsilon \sum_i c_i \chi^i), \;\; 0 < \epsilon \ll 1.$$

Two distinct cases (All you need to know!)

- First order model (sustained and decaying)

$$\begin{aligned} \sigma(\chi) &= \sigma_0(1 + \epsilon\chi), \\ G(\chi) &= -\sigma_0(\chi + \frac{1}{2}\epsilon\chi^2). \end{aligned}$$

- Second order model (sustained)

$$\begin{aligned} \sigma(\chi) &= \sigma_0(1 - \epsilon\chi^2) \\ G(\chi) &= -\sigma_0(\chi - \frac{1}{3}\epsilon\chi^3) \end{aligned}$$

First order model

Nonlinear G-S equation (full and truncated)

$$\begin{aligned}\Delta^*\chi+\sigma_0^2(1+\frac{3}{2}\epsilon\chi)\chi &= 0.\\ \epsilon\sigma_0^2\left\langle\chi_1^3\right\rangle\alpha_1^2+(\sigma_0^2-\sigma_1^2+2\epsilon\sigma_0^2\left\langle\chi_0\chi_1^2\right\rangle)\alpha_1 & \\ +\sigma_0^2\left\langle\chi_0\chi_1\right\rangle+\epsilon\sigma_0^2\left\langle\chi_0^2\chi_1\right\rangle &= 0.\end{aligned}$$

- near resonance: $\sigma_0 \approx \sigma_1$.

$$\alpha_1=\pm\sqrt{-\frac{\langle\chi_0\chi_1\rangle}{\langle\chi_1^3\rangle}\frac{1}{\epsilon}}+o(\epsilon^0)$$

- away from resonance: $\sigma_0^2-\sigma_1^2 \sim o(\epsilon^0)$

$$\begin{aligned}\alpha_1^l &\approx -\frac{\sigma_0^2-\sigma_1^2}{\epsilon\sigma_0^2\left\langle\chi_1^3\right\rangle}\leftarrow \text{large amp. root}\\ \alpha_1^s &\approx -\frac{\sigma_0^2\left\langle\chi_0\chi_1\right\rangle}{\sigma_0^2-\sigma_1^2}\leftarrow \text{small amp. root}\end{aligned}$$

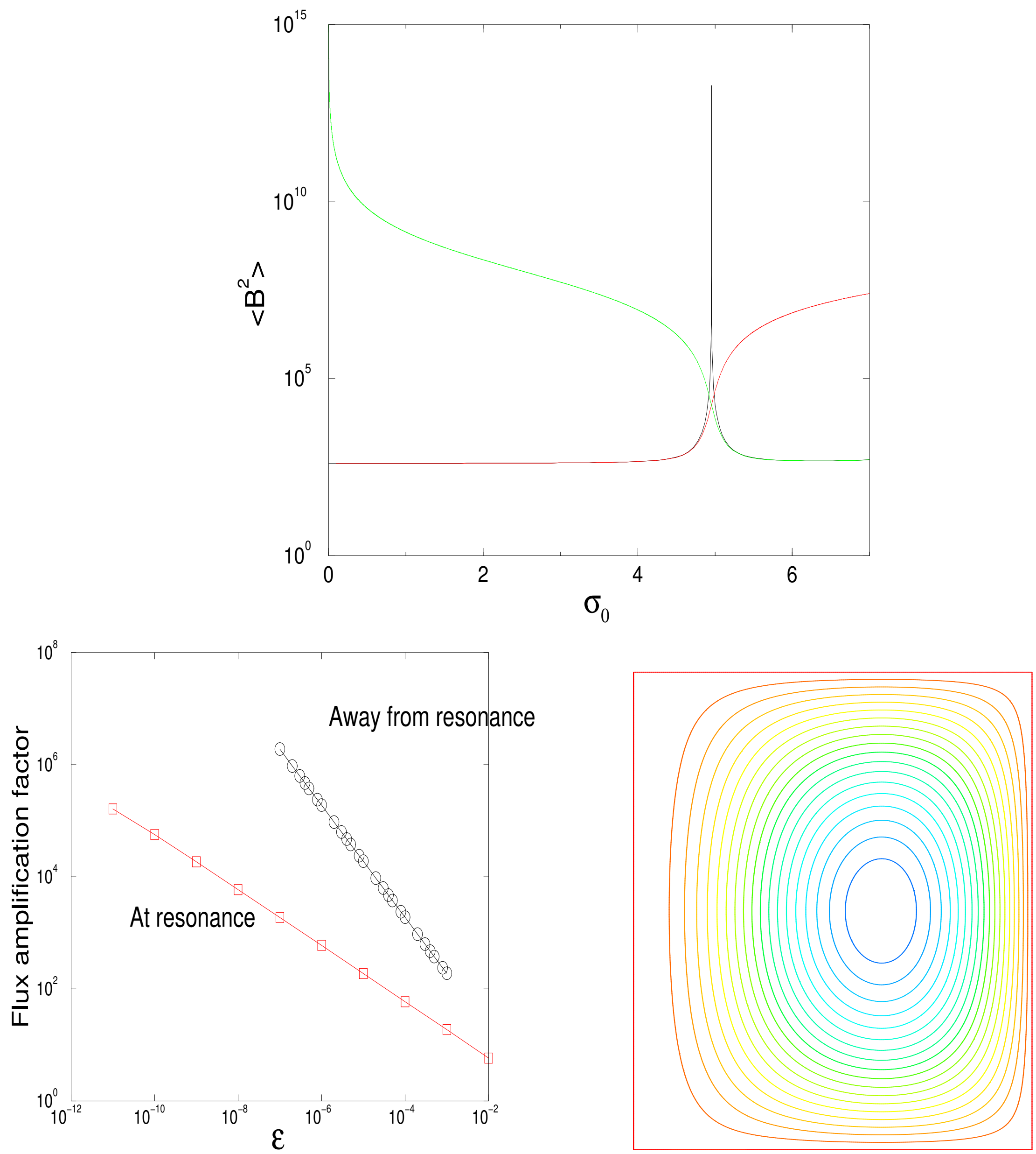

Figure 2: Top: Magnetic energy of two branches of solution versus σ_0. Bottom left: flux amplification factor versus ϵ scaling at and away from the resonance. Bottom right: flux contours of red branch solution at $\sigma_0 = 5.5$ (no flip!)

Second order model

Nonlinear G-S equation (full and truncated)

$$
\begin{aligned}
\Delta^* \chi + \sigma_0^2 \chi (1 - \epsilon \chi^2)(1 - \frac{1}{3}\epsilon \chi^2) &= 0. \\
-\epsilon \sigma_0^2 \langle \chi_1^4 \rangle \alpha_1^3 - 3\epsilon \sigma_0^2 \langle \chi_0 \chi_1^3 \rangle \alpha_1^2 & \\
+(\sigma_0^2 - \sigma_1^2 - 3\epsilon \sigma_0^2 \langle \chi_0^2 \chi_1^2 \rangle)\alpha_1 & \\
+\sigma_0^2 \langle \chi_0 \chi_1 \rangle - \epsilon \sigma_0^2 \langle \chi_0^3 \chi_1 \rangle &= 0
\end{aligned}
$$

- near resonance: $\sigma_0 \approx \sigma_1$.

$$
\alpha_1 = \frac{\langle \chi_0 \chi_1 \rangle^{1/3}}{\langle \chi_1^4 \rangle} \frac{1}{\epsilon^{1/3}} + o(\epsilon^0)
$$

  only one real root.

- away from resonance: $\sigma_0^2 - \sigma_1^2 \sim o(\epsilon^0)$
  - $\sigma_0^2 < \sigma_1^2$: one real root
  - $\sigma_0^2 > \sigma_1^2$: three branches

Second order model –continued

Away from resonance:

- $\sigma_0^2 < \sigma_1^2$: one real root

$$\alpha_1 \approx -\frac{\sigma_0^2}{\sigma_0^2 - \sigma_1^2} \langle \chi_0 \chi_1 \rangle$$

recovering the linear solution.

- $\sigma_0^2 > \sigma_1^2$:

$$\begin{aligned}
\alpha_1^{(1)} &\approx (\frac{\sigma_0^2 - \sigma_1^2}{\sigma_0^2 \langle \chi_1^4 \rangle})^{1/2} \epsilon^{-1/2}. \\
\alpha_1^{(2)} &\approx -\alpha_1^{(1)} \\
\alpha_1^{(3)} &\approx -\frac{\sigma_0^2}{\sigma_0^2 - \sigma_1^2} \langle \chi_0 \chi_1 \rangle
\end{aligned}$$

$\alpha_1^{(3)}$ recovers the linear result.

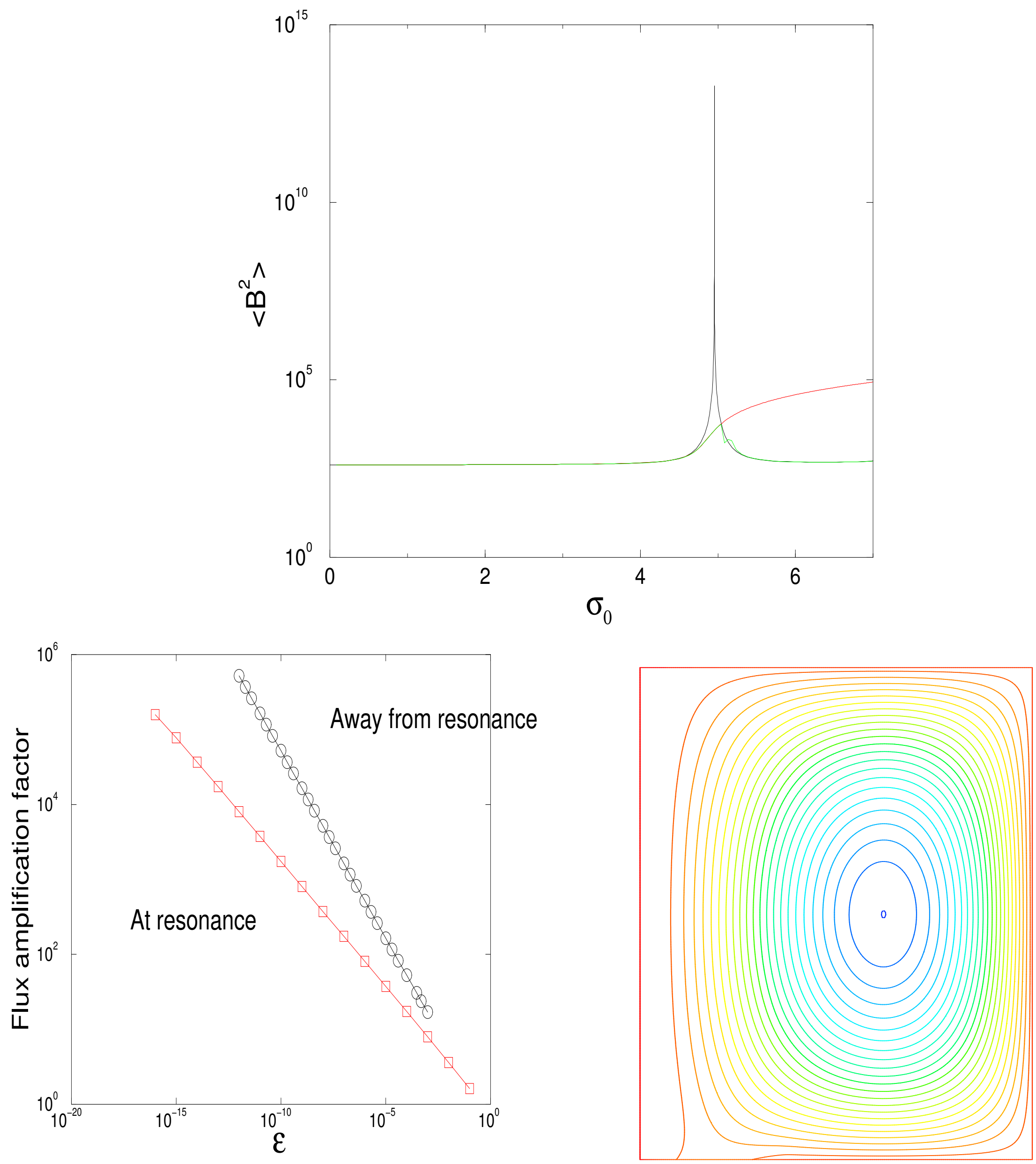

Figure 3: Top: Magnetic energy of distinct branches of solution versus σ_0. Bottom left: flux amplification factor versus ϵ scaling at and away from the resonance. Bottom right: flux contours of red branch solution at $\sigma_0 = 5.5$ (no flip!)

Conclusions

- Force-free: $\mathbf{j} = \sigma \mathbf{B}$ and $\mathbf{B} \cdot \nabla \sigma = 0$.
- Taylor state: σ is a global constant.
- Flux amplification in driven compact toroids depends on the Jensen-Chu resonance.
- In Taylor's theory, accessible magnetic configuration bounded by the first Jensen-Chu resonance.
- Force-free but partially relaxed state (nonlinearity) regularizes Jensen-Chu singularity.
  - Exact agreement between analytics (truncated spectral model) and numerics (full model).
- Additional accessible magnetic configurations.
  - The preferred branch of solution and transition between branches of solution are issues to be settled by dynamics.